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Higher order and boundary scaling fields in the Abelian sandpile model

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Abstract

The Abelian sandpile model (ASM) is a paradigm of self-organized criticality (SOC) which is related to $c = -2$ conformal field theory. The conformal fields corresponding to some height clusters have been suggested before. Here we derive the first corrections to such fields, in a field theoretical approach, when the lattice parameter is non-vanishing, and consider them in the presence of a boundary.

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1. Introduction

Self-organized criticality is believed to be the underlying reason for the scaling laws seen in a number of natural phenomena [1]. Dynamics of a self-organized critical system is such that the system naturally approaches its critical state and exhibits long-range orders and scaling laws. This means that, unlike most statistical models, without any fine-tuning of some parameters such as temperature, the system reaches its critical point.

The concept was first introduced by Bak, Tang and Wiesenfeld [2]. In their paper, they proposed the Abelian sandpile model (ASM) as a model of self-organized criticality. Since then many different models exhibiting the same phenomenon have been developed, but ASM is still the simplest, most studied model, in which many analytical results have been derived. For a good review see [3].

Many exact results are derived in this theory. The first analytical calculation, which paved the road for other analytical results, was done by Dhar [5]. Probabilities of some specific clusters, known as weakly allowed clusters (WACs), were then computed [6]. The simplest of these clusters is the one-site height-1 cluster. The probabilities of other one-site clusters with height above 1 were computed in [7]. Among other analytical results one can mention the results on boundary correlations of height variables and the effect of boundary conditions [8–12], on the presence of dissipation in the model [10, 11, 13], on field theoretical approaches [4, 10, 14, 15], on finite-size corrections [6, 12] and many other results [3].

On the other hand, the model has been related to some other lattice models such as spanning trees, which correspond to the well-known $c = -2$ conformal model. Mahieu and Ruelle [4] related ASM to logarithmic conformal field theory through introducing scaling fields corresponding to the weakly allowed clusters (WACs), such as one-site height-1 cluster. This was done through comparison of correlation functions in the two models, and the fields corresponding to a list of simplest WACs were obtained. Later, Jeng [9] introduced an elegant method to calculate such fields for arbitrary WAC and showed that for all of these clusters the scaling dimension is 2. Meanwhile, a more direct way to show this correspondence was developed in [14]. The benefit of this new method is that it comes from an action and is not found merely by comparing correlation functions. This allows some further investigations that were not possible before. The fields associated with other one-site clusters are also derived [10, 13, 15].

However, most of these results are obtained in the thermodynamic limit; i.e., it is supposed that the size of the system, L, is very large and the lattice spacing, a, is ignorable. Though some results are obtained in situations where these conditions are not satisfied, we would like to use the method introduced in [14] to obtain scaling fields when we are far from the thermodynamic limit. To modify the first assumption ($L \to \infty$) one may consider the finite-size effects and to see the effects of modifying the second condition, one may consider how the scaling fields depend on the lattice spacing. Because of discrete nature of the model, it is important to investigate this problem.

In this paper, we first review the method introduced in [14] to derive scaling fields and then using the method, we derive the higher order corrections to the first order of lattice spacing. At the end, we discuss the effect of the boundary on the field derived using the method of [14].

2. Scaling fields

As mentioned before, a correspondence between ASM and other well-known statistical models has been established, the most significant of which are the connection to $q \to 0$ limit of the q-state Potts model [18], spanning trees [6] and dense polymers [18]. All of these models display conformal symmetry at their critical points and the proposed CFT corresponding to them is $c = -2$ model, which is a logarithmic CFT [19]. Therefore, it is reasonable to seek a straight way to connect ASM to this conformal field theory. One of the first attempts was done by Mahieu and Ruelle. Through some arguments of locality and scaling dimension and comparison of height correlations of ASM with correlation functions of different fields in $c = -2$ model, they noted that certain fields can be associated with a number of clusters; namely, WACs [4]. Though it was a great step forward, the method still had the shortcoming that it was based only on comparison of correlation functions. Inspired by their results and a method suggested by Ivashkevich to relate dense polymers and CFT [16]—later elaborated for all trees and forests by Caracciolo *et al* [17]—Moghimi-Araghi, Rajabpour and Rouhani solidly confirmed the correspondence between $c = -2$ and ASM height correlations [14]. They deduced the results of Mahieu and Ruelle [4] in a straightforward manner and added some further subtleties to incorporate higher corrections of height correlations in the fields.

2.1. Grassmannian method

In their paper, Moghimi-Araghi, Rajabpour and Rouhani [14] re-expressed the Majumdar–Dhar probability of clusters in the bulk in terms of a contrived statistical system according to

Berezin's definition of Grassmannian integrals:

$$P(C) = \frac{\det \Delta'_c}{\det \Delta} = \frac{\int \mathrm{d}\theta_i \, \mathrm{d}\bar{\theta}_j \exp\left(\sum \theta_i \Delta'_{ij} \bar{\theta}_j\right)}{\int \mathrm{d}\theta_i \, \mathrm{d}\bar{\theta}_j \exp\left(\sum \theta_i \Delta_{ij} \bar{\theta}_j\right)} = \frac{\int \mathrm{d}\theta_i \, \mathrm{d}\bar{\theta}_j \exp\left(\sum (\theta_i \Delta_{ij} \bar{\theta}_j + \theta_i B_{ij} \bar{\theta}_j)\right)}{\int \mathrm{d}\theta_i \, \mathrm{d}\bar{\theta}_j \exp\left(\sum \theta_i \Delta_{ij} \bar{\theta}_j\right)}. \tag{1}$$

Here, Δ is the lattice Laplacian matrix and B is the defect matrix defined in [6].

Evidently, since $[\theta_i\bar{\theta}_j, \theta_k\bar{\theta}_l] = 0$ for all i, j, k and l, and hence, one can use Baker–Hausdorff–Campbell formula, the above expression can be interpreted as the expectation value of a field,

$$\varphi_c = \exp\left(\sum \theta_i B_{ij} \bar{\theta}_j\right). \tag{2}$$

Subtleties arise in transition to the continuum limit; one can perform the transition in different ways which yield different results. Naïvely, one can find the continuum limit of $\Sigma\theta_i B_{ij}\bar{\theta}_j$ in the exponent by expanding θ and $\bar{\theta}$ around θ_0. As an example, if we take the cluster to be a single height-1 site, the corresponding field obtained with this method turns out to be

$$\sum \theta_i B_{ij} \bar{\theta}_j|_{S=S_0} \propto \partial\theta\bar{\partial}\bar{\theta} + \bar{\partial}\theta\partial\bar{\theta} - \frac{1}{4}\theta\bar{\theta}. \tag{3}$$

As θ and $\bar{\theta}$ are Grassmann variables, the resulting exponential can be calculated easily, since its Taylor series ends at quadratic terms in θ and $\bar{\theta}$. The fields obtained in this manner are not in accord with results of Mahieu and Ruelle [4], though there exist some similarities.

The second more careful method of transition to continuum is to expand the exponential in terms of θ and $\bar{\theta}$ first, and expanding θ and $\bar{\theta}$ around θ_0 thereafter. As an illustration, suppose we want to find the field corresponding to a height-1 site in the bulk [14]. This can be sought through the correlation function of two height-1 fields in the bulk in Majumdar–Dhar method [6] and relating that to the fictitious statistical system:

$$\begin{aligned} P(C_1, C_2) = \det(1 + GB) &= \frac{\int \mathrm{d}\theta_i \, \mathrm{d}\bar{\theta}_j \exp\left(\sum \left(\theta_i \Delta_{ij} \bar{\theta}_j + \theta_i B^1_{ij} \bar{\theta}_j + \theta_i B^2_{ij} \bar{\theta}_j\right)\right)}{\int \mathrm{d}\theta_i \, \mathrm{d}\bar{\theta}_j \exp\left(\sum \theta_i \Delta_{ij} \bar{\theta}_j\right)} \\ &= \left\langle \exp\left(\theta_i B^1_{ij} \bar{\theta}_j\right) \exp\left(\theta_i B^2_{ij} \bar{\theta}_j\right)\right\rangle. \end{aligned} \tag{4}$$

Here, G is the inverse of the matrix Δ.

If we calculate $\langle \phi_{C_1}\phi_{C_2}\rangle$ using Wick theorem, we find different terms. One can classify them according to the number of long-range contractions between Grassmann variables of the two fields—this number is always even, as a contraction of θ of C_1 to $\bar{\theta}$ of C_2 must be compensated with a contraction of $\bar{\theta}$ of C_2 with θ of C_1. Having no long-rang contractions simply reveals the probabilities of the clusters individually and says nothing about the correlation of the two fields. The first relevant order is when we have two long-range contractions and contract the other Grassmann variables within their own clusters. So, to obtain the scaling field, one can contract all θ's and $\bar{\theta}$'s leaving a pair of them uncontracted. The resulting field is of the form $\phi = \sum \theta_i A_{ij} \bar{\theta}_j$. Now, expanding all θ's and $\bar{\theta}$'s around θ_0 to the second order, one obtains the desired scaling field; for instance, for a one-site cluster with height 1, we have

$$\phi_{S_0}(z) = -\frac{4(\pi - 2)}{\pi^2} :\partial\theta\bar{\partial}\bar{\theta} + \bar{\partial}\theta\partial\bar{\theta}: . \tag{5}$$

which is the same field obtained in [4]. This expansion is only up to the leading term in a/r, with a and r being lattice spacing and the typical distance between the two scaling fields respectively. In the following section, we consider higher order terms and derive corrections to these scaling fields.

3. Calculation of higher orders

Obtaining the scaling fields in this way is a wearisome belabored task and is prone to be afflicted by human error. Thus, we developed a Mathematica [20] code by which we could handle these calculations fast, reliable and adjustable for different configurations. We checked the code by recalculating the fields assigned with simple WACs; the final results were in accordance with those of Moghimi-Araghi *et al* [14].

Now we would like to see what fields would arise if we consider higher order terms in a/r. The configuration probability, $P(C)$, and the correlation function, $P(C_1, C_2)$, are the only significant data we have at hand to figure out the form of the corresponding scaling field. Following the Grassmannian method, one can consider the expression $\exp(\theta B\bar{\theta})$ as the field associated with a WAC. First we expand this expression in terms of θ and $\bar{\theta}$. In the Grassmannian method we contract all the fields except two of them, but we would like to derive several different terms depending on how many θ's and $\bar{\theta}$'s are left uncontracted. The resulting field will have the form

$$\phi_C = c_0\mathbf{I} + \sum \mathcal{A}^{(1)}_{ij}\theta_i\bar{\theta}_j + \sum \mathcal{A}^{(2)}_{ijkl}\theta_i\bar{\theta}_j\theta_k\bar{\theta}_l + \cdots + \sum \mathcal{A}^{(n)}_{i_1 i_2\cdots i_{2n}}\theta_{i_1}\bar{\theta}_{i_2}\cdots\theta_{i_{2n-1}}\bar{\theta}_{i_{2n}} + \cdots. \tag{6}$$

Note that the above series terminates due to the Grassmann nature of θ and $\bar{\theta}$. In each term n is the number of uncontracted pairs, and these pairs, when contracted with a similar pair of another field produce long-range correlations. To derive the coefficient $\mathcal{A}^{(n)}$ one can use a generalized version of graphical method used in [14], yet it is possible to find these coefficients in an easier way. Contraction of all pairs will give us the probability of the cluster which is $\langle\phi_C\rangle = P(C) = \det(I + BG)$. It is easy to see that the coefficient of G_{ij} in $P(C)$ is $A^{(1)}_{ij}$ and the coefficient of $G_{ij}G_{kl}$ is $\mathcal{A}^{(2)}_{ijkl}$. In this manner, we can find all the needed coefficients, $\mathcal{A}$'s, in ϕ_C easily; that is, first, we compute $P(C) = \det(I + GB)$ as a function of G_{ij}'s. Second, we derive the coefficient of G_{ij}, for all i and j to find $\mathcal{A}^{(1)}_{ij}$ or derive the coefficient of $G_{ij}G_{kl}$ to find $\mathcal{A}^{(2)}_{ijkl}$. Then remains just one additional step: to formally expand θ_i's and $\bar{\theta}_i$'s around θ_0, where θ_0 is some point inside the cluster. As an example, doing all the procedure explained above we arrive at the following expression for the one-site height-1 cluster up to $O(a^4)$:

$$\phi_{S_0} = a^2\frac{4(2-\pi)}{\pi^2}(\partial\theta\bar{\partial}\bar{\theta} + \bar{\partial}\theta\partial\bar{\theta}) + a^4\frac{2(2-\pi)}{3\pi^2}\Bigg(\partial\theta\partial^3\bar{\theta} + \partial^3\theta\partial\bar{\theta} + \bar{\partial}\theta\bar{\partial}^3\bar{\theta} + \bar{\partial}^3\theta\bar{\partial}\bar{\theta} - \frac{3}{2(2-\pi)}(\partial^2+\bar{\partial}^2)\theta(\partial^2+\bar{\partial}^2)\bar{\theta}\Bigg) + a^4\frac{32(2-\pi)}{\pi}\partial\theta\partial\bar{\theta}\bar{\partial}\theta\bar{\partial}\bar{\theta}. \tag{7}$$

In simplifying the original terms of the field, the evolution equations of $c = -2$ action were applied to eliminate some terms:

$$\partial\bar{\partial}\theta = \partial\bar{\partial}\bar{\theta} = 0. \tag{8}$$

The first line of equation (7) is the ordinary field derived before. The second line is the expansion of $\mathcal{A}^{(2)}_{ij}$ up to $O(a^4)$, and the third line originates from the term $\mathcal{A}^{(4)}_{ijkl}$. Note that there are no terms of the order a^3. In fact there were some terms, but all vanish due to the equation of motion (equation (8)). The fields of the order of a^4 are irrelevant under renormalization group and so could be neglected in the scaling limit. But since in ASM, the size of lattice spacings may not be neglected in general, these extra terms could be relevant in some calculations. Note that the correction to the leading term is not unique and may depend on some of small scale properties of the model such as the lattice. We have considered a square lattice in our calculations.

Though we have derived the higher corrections to the scaling field, yet it is not the end of the story. If we are going to take care of terms of the order of a^4 in scaling fields, we have to treat in the same way with the action. The $c = -2$ action is obtained if we collect only the terms up to the second order of a. So, if the size of the lattice is important, the action admits some modification. Changing the action will change the Green function and we already know that in the calculation of higher order correction to correlation functions, one has to take into account the higher order terms of discrete Laplacian's Green function. This can be done readily in our scheme: we have the discrete version of the action, just we have to expand it up to order a^4. The result is

$$A = \frac{1}{\pi}\int\left(\partial\theta\bar{\partial}\bar{\theta} + a^2\left(\frac{1}{12}(\partial^2\theta\partial^2\bar{\theta} + \bar{\partial}^2\theta\bar{\partial}^2\bar{\theta}) + \frac{1}{2}\partial\bar{\partial}\theta\partial\bar{\partial}\bar{\theta}\right)\right). \tag{9}$$

Note that the new action is still quadratic and hence integrable; however it does not have conformal symmetry, although the off-critical term is an irrelevant term and vanishes under RG. But we consider the problem when the small scale of the system cannot be neglected completely, this forces us to keep the extra, and irrelevant, terms in the action. The last point we would like to mention is that the a^3 terms in the scaling field still vanish, since the application of the equation of motion terms of the order of a^5 only.

With all these corrections, we are ready to calculate the correlation function of the field associated with the cluster S_0 to the order of a^6. We should consider both changes to the field and action, but as we only consider the leading terms, the calculation is not that hard: the changes to Green functions should be taken into account only when you compute the correlation of the field in the first line of equation (7) and there is no need to compute correlations of the two fields of the order of a^4. The result is

$$\langle\phi_{S_0}(0)\phi_{S_0}(\vec{r})\rangle = -[P(S_0)]^2\frac{a^4}{r^4}\left[\frac{1}{2} + \frac{a^2}{r^2}\left(\frac{1}{3} + \frac{1}{\pi^3 P(S_0)}\right)\right]. \tag{10}$$

As you see, the correlation is still rotational invariant. This is because we do not have a preferred direction. In other words, you cannot find a vector to produce a 'dipolar' effect. But if we go to higher order terms, because of 'quadrapole' terms (tensor terms) one arrives at correlations that do not preserve rotational symmetry.

At the end of this paper, we would like to express where these different corrections come from by comparing it with the procedure introduced in [9]. In the scheme of [9], one can easily identify the approximations used to derive the scaling fields. First, the long-range Green function is kept only up to the logarithmic term. Second, in computing the determinant assigned to the probability of occurrence of WACs, such as that in the nominator of equation (4), only one off-diagonal term (that contains the long-range Green function) is picked. Third, Green functions such as $G(\vec{r} + \vec{u})$, with $\vec{u}$ being of the order of lattice spacing, are expanded only to second order of lattice spacing. The corresponding approximations in the scheme used in [14] are: (i) While calculating the continuum limit of the action, just keep terms of the order of a^2. (ii) Keep only one pair of θ and $\bar{\theta}$ uncontracted. (iii) Expand θ's and $\bar{\theta}$'s appearing in the resulting field to the order of a^2.

To derive the next order correlation functions, one should replace all the above three considerations with those that take into account higher order terms. Replacing the first one leads to changing the action of the theory (equation (9)). In the second approximation only terms with one uncontracted pair were taken into account, so for the next leading term we have also kept the terms with two uncontracted pairs. The result is appearing of a field appears with four Grassmann variables in the third line of equation (7). To modify the third approximation, we have expanded the θ variables to the order of a^4 as you can see in the second line of equation (7).

4. Fields near a boundary

In this section, we would like to derive the properties of scaling fields in ASM when a boundary is present at the theory. This will help us to derive finite-size properties and surface critical properties of the model. In the context of the conformal field theory, the boundary is usually taken to be the real line in the complex plane and the system is supposed to fill the upper half-plane. The correlation functions of scaling fields in this geometry reveal surface critical exponents and the finite-size scaling properties. It is derived that under certain boundary conditions the correlation functions of scaling fields in this geometry are the holomorphic part of bulk correlation functions of same fields together with their images (which live in the lower half-plane) [21]. The idea was later generalized to the case of logarithmic conformal field theory [22, 23].

Many properties of ASM in the presence of such boundary have been derived. Ivashkevich calculated all two-point functions of all height variables along closed and open boundaries [8]. Using these correlation functions, Jeng identified the height variables with certain fields in $c = -2$. A more detailed treatment is given in [15]. However, this identification is by examining the correlation functions. We would like to apply the Grassmaniann method to the ASM with the mentioned boundary. Note that as this method is only for WACs, we are not able to derive the height-2 or more one-site fields.

The question we are going to address is that if we consider the model near a boundary, the toppling matrix, Δ, and the Green function G change. Since in the derivation of the fields using the Grassmaninam method, we use the Green function, the resulting field in the presence of a boundary may be different from the field in the bulk. We would like to emphasize that this section is somehow a consistency check. We know that the field near a boundary should be the same as the bulk ones, but it is not clear in the context of the Grassmaniann method, as the Green function is different in this case. However, we show that the method reveals the same field and is consistent.

To begin with, we would like to calculate the one-point function of a WAC in the upper half boundary. This is given by the following expression:

$$P(C) = \frac{\det \Delta'_c}{\det \Delta} = \frac{\int \mathrm{d}\theta_i\, \mathrm{d}\bar{\theta}_j \exp\left(\sum \theta_i \Delta'_{ij} \bar{\theta}_j\right)}{\int \mathrm{d}\theta_i\, \mathrm{d}\bar{\theta}_j \exp\left(\sum \theta_i \Delta_{ij} \bar{\theta}_j\right)} = \frac{\int \mathrm{d}\theta_i\, \mathrm{d}\bar{\theta}_j \exp\left(\sum (\theta_i \Delta_{ij} \bar{\theta}_j + \theta_i B_{ij} \bar{\theta}_j)\right)}{\int \mathrm{d}\theta_i\, \mathrm{d}\bar{\theta}_j \exp\left(\sum \theta_i \Delta_{ij} \bar{\theta}_j\right)}, \tag{11}$$

which is the same expression as in the bulk, just you should use the appropriate matrix Δ. Again B^C is the matrix defined in the Majumdar–Dhar method for the specified WAC, C. So it is seen that the field associated with this cluster is again $\exp \theta_i B^C_{ij} \bar{\theta}_j$, but we should keep in mind that to derive the proper form of the field we use boundary Green functions which are different from the bulk's Green functions. Let us expand this expression and contract *all* θ's and $\bar{\theta}$'s to find the probability of the cluster. Note that as we would like to find a one-point correlation function, we do not leave two of the variables uncontracted. The Green function of the theory with this geometry is obtained easily by the method of images:

$$\begin{aligned} G_{\mathrm{op}}(\vec{r}_1, \vec{r}_2) &= G(x, y_1 - y_2) - G(x, y_1 + y_2), \\ G_{\mathrm{cl}}(\vec{r}_1, \vec{r}_2) &= G(x, y_1 - y_2) + G(x, y_1 + y_2 - 1), \end{aligned} \tag{12}$$

where $G_{\mathrm{op/cl}}$ is the Green function open/closed boundary conditions and G is the Green function in the bulk. Also $\vec{r}_i = (x_i, y_i)$ and $x = x_2 - x_1$.

So, each contraction of θ variables has two terms; one, which we call short-range (SR) contraction, comes from the first terms in the above equations and is the bulk Green function

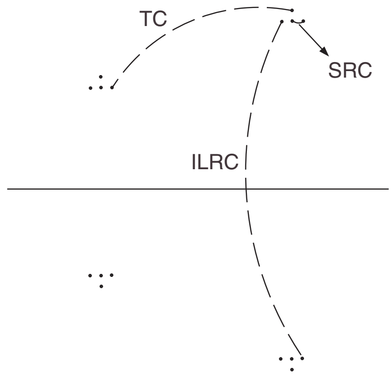

Figure 1. Examples of SR-, ILR- and T-contractions near a boundary.

of the points $\vec{r}_1$ and $\vec{r}_2$. The other one, which we call image-long-range (ILR) contraction, is the bulk Green function of the point $\vec{r}_1$ with the image of the point $\vec{r}_2$ (see figure 1).

Take a typical expression of θ's and $\bar{\theta}$'s to be fully contracted. We can classify the terms appearing in the result by number of ILR contractions (ILRCs), n_I. If n_I is set to zero, we will arrive at the bulk probability of the cluster C. The first correction due to the presence of the boundary appears when $n_I = 1$. This means that we should contract all the variables in the way we did in the bulk except two which we will contract later using a ILR contraction. The procedure reduces to that discussed by [14] and for example the result for the height-1 one-site cluster would be given by equation (5), only you have to contract the remaining θ and $\bar{\theta}$ using ILRC. In the scaling limit this gives

$$\langle \phi_{S_0}(\vec{r}) \rangle_{\text{op/cl}} = P(1)\left(1 \mp \frac{1}{4y^2} + \cdots\right), \tag{13}$$

where y is the distance of the field from the boundary and the minus/plus sign corresponds to the closed/open boundary condition which is consistent with previous results. Also if we take a typical WAC, with the corresponding field [4, 9]

$$\phi_S = -[A_S : \partial\theta\bar{\partial}\bar{\theta} + \bar{\partial}\theta\partial\bar{\theta} : +B_{1S} : \partial\theta\partial\bar{\theta} + \bar{\partial}\theta\bar{\partial}\bar{\theta} : +\mathrm{i}B_{2S} : \partial\theta\partial\bar{\theta} - \bar{\partial}\theta\bar{\partial}\bar{\theta} :], \tag{14}$$

it is easy to check that the second and third terms of this field do not give any contribution when they are ILR-contracted. Hence the one-point boundary correlation function is obtained to be

$$\langle \phi_S \rangle_{\text{op/cl}} = P(S) \mp \frac{A_S}{4y^2}, \tag{15}$$

just as is indicated in [9].

Now we move on to calculate the field of a WAC explicitly. Suppose you have two WACs C_1 and C_2 and would like to compute the probability of such configuration. Again we have an expression such as (4). Now we can classify them depending on the number of trans-contractions (TCs), n_T and the number of ILRCs among the inter-contractions, n_I. Setting both n_T and n_I equal to zero, we will arrive at $P(C_1)P(C_2)$ with $P(C)$ being the bulk probability of the cluster C. This term together with the term coming from $n_T = 0$ and $n_I = 1$ reveals $\langle \phi_{C_1}(\vec{r}_1) \rangle_{\text{op/cl}} \langle \phi_{C_2}(\vec{r}_2) \rangle_{\text{op/cl}}$, the disjoint boundary probability of the two clusters. To find the correlation of the two clusters, we should set n_T nonzero. Taking $n_T = 2$ (the smallest nonzero value for n_T) and $n_I = 0$ the problem reduces to that in the bulk: contract all θ's and $\bar{\theta}$'s using the *bulk* Green function except two, which are left to be contracted with the θ variables of the other cluster's field. This means that the field derived in this manner is exactly the same as that in the bulk and we do not need to care about the fact that the Green functions have changed.

The first terms that contain both joint probabilities and the effect of boundary Green function appear when we set $n_T = 2$ and $n_I = 1$. This means that you should contract all θ's and $\bar{\theta}$'s but four, using SRC, two of remaining will be contracted with the variables of other fields and two will be ILR contracted. Such a field has been calculated in the previous section and we have seen it is of the order of a^4 and vanishes in the scaling limit. So, if we consider the problem in the scaling limit, the Grassmaniann method says the boundary field is just the same as the bulk field. However, this does not mean that the effect of the boundary could be neglected completely, because in the trans-contractions we should use boundary Green functions.

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References

[1] Jensen H J 1998 *Self-Organized Criticality* (Cambridge: Cambridge University Press)
[2] Bak P, Tang C and Wiesenfeld K 1987 *Phys. Rev. Lett.* **59** 381
[3] Dhar D 1999 *Preprint* cond-mat/9909009
[4] Mahieu S and Ruelle P 2001 *Phys. Rev.* E **64** 066130 (*Preprint* hep-th/0107150)
[5] Dhar D 1990 *Phys. Rev. Lett.* **64** 1613
Dhar D 1990 *Phys. Rev. Lett.* **64** 2837
[6] Majumdar S N and Dhar D 1992 *Physica* A **185** 129
[7] Prizzhev V B 1994 *J. Stat. Phys.* **74** 955
[8] Ivashkevich E V 1994 *J. Phys. A: Math. Gen.* **27** 3643
[9] Jeng M 2005 *Phys. Rev.* E **71** 016140 (*Preprint* cond-mat/0407115)
[10] Jeng M 2005 *Phys. Rev.* E **71** 036153 (*Preprint* cond-mat/0405594)
[11] Jeng M 2004 *Phys. Rev.* E **60** 051302 (*Preprint* cond-mat/0312656)
[12] Ruelle P 2002 *Phys. Lett.* B **539** 172 (*Preprint* hep-th/0203105)
[13] Piroux G and Ruelle P 2004 *J. Stat. Mech.* 0410 P005 (*Preprint* hep-th/0407143)
[14] Moghimi-Araghi S, Rajabpur M A and Rouhani S 2005 *Nucl. Phys.* B **718** [FS] 362 (*Preprint* cond-mat/0410434)
[15] Jeng M, Piroux G and Ruelle P 2006 *J. Stat. Mech.* 0610, P015 (*Preprint* cond-mat/0609284)
[16] Ivashkevich E V 1999 *J. Phys. A: Math. Gen.* **32** 1691
[17] Caracciolo S, Jacobsen J L, Saleur H, Sokal A D and Sportiello A 2004 *Phys. Rev. Lett.* **93** 080601 (*Preprint* cond-mat/0403271)
[18] Saluer H and Duplantier B 1987 *Phys. Rev. Lett.* **58** 2325
Coniglio A 1989 *Phys. Rev. Lett.* **62** 3054
[19] Gurarie V 1993 *Nucl. Phys.* B **410** 535–49 (*Preprint* hep-th/9303160)
[20] Wolfram S 2005 *Mathematica Ver. 5.1* (Wolfram Research)
[21] Cardy J L 1984 *Nucl. Phys.* B **240** [FS12] 514
[22] Moghimi-Araghi S and Rouhani S 2000 *Lett. Math. Phys.* **53** 49 (*Preprint* hep-th/0002142)
[23] Kogan I I and Wheater J F 2000 *Phys. Lett.* B **486** 353 (*Preprint* hep-th/0003184)